The Problem of Inconsistency of Arithmetic

M. George
Dept. of Physics
Southwestern College
Chula Vista, CA 91910
5/10/2015

Introduction

Truth is elusive. It is a web of meaning and relationships that goes far beyond each of our individual lives, our individual circumstances, the limited anthropomorphic conceptions in which we clothe the universe, and our times. There is much uncertainty about truth, just as there is much uncertainty and risk involved in human life. Perceived truths can be fragile. In physics, we recognize, for example, the apparent incompatibility of the theory of relativity and that of quantum mechanics. Could this mean that these viewpoints are irreconcilable, and that fundamental physics, for example, is presenting us with truths about the world that present both an ambiguous and inconsistent picture of reality? To most physicists, it merely implies that both theories cannot be correct, but I do not think that we should summarily dismiss the possibility that truth may not be unitary, that it may be entwined in uncertainties and ambiguities, and that it may present us with irreconcilable inconsistencies. We formulate rules about reasoning, but these are merely arbitrary pebbles thrown into a huge pond, and the tiny waves reach out but do not encompass the greatness of truth.

In this context, is it possible that arithmetic is inconsistent, i.e. that we can arrive at two distinct results that contradict one another? Gödel's work [1] in metamathematics indicates that ordinarily, because the arithmetic we use is a sufficiently rich computational system, we cannot establish, within the confines of that system itself, that it is consistent. Other axiom systems are also not going to radically alter this situation, since they are usually subject to Gödel's conclusions as well, and, being insufficient in themselves to provide a consistency proof, for such a system in itself, can hardly be expected to provide an absolute guarantee of consistency for arithmetic.

There are several responses to this. First, we use arithmetic of varying complexity, and it is perfectly within the realm of possibility that in some extremely complicated mathematical derivations or computations, some subtle inconsistencies might arise, that would be totally irrelevant for the more mundane uses of arithmetic. Secondly, it seems quite apparent that if we use arithmetic in some naïve way, and arrive at an inconsistency, we should be able to remedy this by using a more sophisticated approach, and perhaps limiting definitions appropriately to avoid inconsistency. Thus, it seems that if we use arithmetic carefully, in routine applications, we can detect and eliminate the possibilities of inconsistencies. From a mathematical point of view, then, such a fundamental philosophical question like consistency seems to be largely irrelevant for the working mathematician.

Despite this last conclusion or possibility, we investigate inconsistency in arithmetic in this paper. My goal here is to present "inconsistency" for arithmetic first at a level of naïve computation, and then gradually progress to subtler levels of inconsistency. It should come as no surprise that a naïve computation (adhering closely to the "rules" of arithmetic, without trying to "trick" anyone) might yield inconsistencies. Mathematics is very subtle, and can easily present us with seeming contradictions which are possible to resolve. As we progress, and “correct” the inconsistencies, I will explain the meaning of the computations, and exactly why the inconsistencies arise. Then, when the development is complete, I will discuss consequences of the final result.

A Naïve Computation

Consider the computation:

1 $$-1 = (-1)^{3/3} = ((-1)^3)^{1/3} = (-1)^{1/3}$$

We obtain nothing surprising here. We can view taking the third root as inverting the function

2 $$y = x^3$$

This has an inverse relation that is also a function. Therefore, there is no complicating factor in this case.

Now, we carry through with the same type of computation, but involving the fourth root:

3 $$-1 = (-1)^{4/4} = ((-1)^4)^{1/4} = 1^{1/4}$$

Naively, this looks contradictory, but the fourth root derives from the inverse relation for

4 $$y = x^4$$

This inverse relation does not define a function, rather a multi-valued relation: The fourth root of 1 in this sense (restricted to real values) can equal -1 or 1. We obtain an ambiguity when we take the fourth root. Thus, choosing the correct option does not lead to any inconsistency here, either. The ambiguity arises because when we take the fourth power of -1 we lose information. Specifically, we lose information about the sign of the original number, as the fourth power of a real number must be non-negative.

What we see in these naïve computations is that we must be very careful with the inverse processes, of extracting roots, because they may not relate to functions, but, instead, multi-valued relations. Indeed, of course, to be very careful, we should use complex numbers.

A Naïve Complex-Number Computation

Let’s start with complex-number computations with a naïve computation as well:

5 $$(-1)^{0.7} = (e^{i\pi})^{0.7} = e^{0.7\pi i} = \cos(0.7\pi) + i \sin(0.7\pi)$$

The choice of 0.7 is arbitrary. Many numbers would be equally satisfactory.

Now, just as with our prior computations, we introduce a subtlety:

6 $(-1)^{0.7} = (-1)^{7/10} = ((-1)^7)^{1/10} = (-1)^{1/10}$

From our prior experience with the even root, one-fourth, we recognize the presence of ambiguity again. Thus, while this looks like a potential inconsistency, again we must consider multi-valued inverses: The function

7 $y = x^{10}$

has a multivalued inverse as a function of complex values. We can capture all of these values for $(-1)^{1/10}$ by writing -1 as

8 $-1 = e^{i\pi + 2n\pi i}$,

where we allow n to be any integer. Then,

9 $(-1)^{1/10} = e^{0.1\pi i + 0.2n\pi i} = \cos(0.1\pi + 0.2n\pi) + i \sin(0.1\pi + 0.2n\pi)$

Comparing (9) with (5), we see that by paying careful attention to the ambiguity introduced in extracting the tenth root, that, once again, a seeming inconsistency is resolved, for example, by taking n = 3.

A More Interesting Computation

Now, we know what the game is. Since we are working with multivalued inverses, we need to pay attention to the ambiguity they entail. That's it: There are no real inconsistencies. Or are there?

Notice, that although we are not obtaining inconsistencies here, there is something troublingly complicated about having to keep track of all of the "branches", specified by each integer value of n above.

Now, we carry out a similar calculation to the above, writing:

10 $(-1)^{0.7} = (-1)^{70/100} = ((-1)^{70})^{1/100} = 1^{1/100}$

Here, we see that again this is just a matter of picking out the right branch: We have lost information about -1 and just made things a little more complicated. We need to express 1 in terms of all possible branches:

11 $1^{1/100} = (e^{2n'\pi i})^{1/100} = e^{2\pi n' i/100} = e^{0.02n'\pi i}$

where n' is some integer. It is clear that n' = 35 will work.

Therefore, we have proved that

12 $(-1)^{0.7} = (-1)^{70/100}$,

with a “probability” of 1/100 (since we have 100 branches to sort through), which is much smaller than the simpler case we considered before. We have arrived at an inconsistency in ordinary arithmetic in the other 99 possibilities involved with the ambiguity. The ambiguity is easy to resolve, but there is an uncertainty in choosing the correct branch.

Is Uncertainty of Interest?

The inconsistencies arise as a consequence of choosing the wrong branch in an “inverse” problem. Taking a root is an inverse process relative to the rather more straightforward process of raising something to a power. This is an artificial separation, because, as we have seen, raising something to a power can be implicated with taking a root.

Computation is a process. Each part of a computation can be represented as a step along a path in a web of possible computational steps. We know, within the context of mathematics, that there are often many possible paths between a starting point and a conclusion. We can represent a particular path in terms of graph theory: (V0, V1, V2, … VN). These VI’s represent the starting and end points for an individual step in our computation or reasoning process, and are vertices of this web of relationships possible within the allowed rules of our mathematical system. (We should also include edges, but for the sake of simplicity, we are not.) If no information is lost or gained in a step, and the step is reversible, then we can feel relatively confident about our derivation. However, with respect to inverse processes as above in extracting roots, a step is not necessarily reversible and may introduce ambiguity that we are going to have to sort through in the end. This results from loss of information, but if we are careful, we can sort through the ambiguities.

However, we introduced a complicating feature, and one that naively appears to be not disruptive. In the sequence of the path of computation of our reasoning, we replaced vertex VK, for some K, by a thoroughly equivalent vertex VK’: we replaced 0.7 by 7/10 or 70/100. In ordinary arithmetic, how could we question the validity of such a replacement? Within certain arithmetic processing, such as machine processing, we know this it is not necessarily the case that replacing 0.7 by 7/10 will not disturb the computation, but here we are just working within the confines of ordinary arithmetic.

Then, the path up to this point becomes (V0, V1, V2, … V(K-1), VK’). Now, in terms of processing, using 70/100 instead of 0.7, or starting from VK’ instead of VK, allows us to separate 70 from 100. But this was a new step, and a different step, V(K+1)’, instead of V(K+1). We thus generate a different path: (V0, V1, … V(K-1), VK’, V(K+1)’, … VN’). And, the endpoint, VN’, turned out to represent a high level of ambiguity relative to the original VN, and this introduces some uncertainty in choosing the correct branch.

Although the ambiguity is always resolvable, the general method we are using is one that cannot be dealt with, because it is outside the system of arithmetic, and does introduce uncertainty. Therefore, even if one removes the inconsistency in this case, we have opened the door to finding others by accident.

Consequences of an Inconsistency

First, it is necessary to address the fears and anxieties entailed in facing something that is startling and challenges us unpleasantly. Does inconsistency in arithmetic, at a theoretical level, resulting from the "practical" reality of uncertainty, imply, necessarily, the collapse of the rational fabric of our world? This does not seem to be entailed. Inconsistency in ideal arithmetic will not be disturbing. As a real possibility of choosing the wrong branch, we cannot ignore this uncertainty in actual reasoning or computation: Inconsistencies are very possible. I think that one cannot exclude the possibility that this will not reach significantly into the practicalities of each of our lives, as reasoning or computation becomes more difficult and complicated.

The practical uses of mathematics are not merely sustained on the basis of theory. There is an empirical element to mathematics, not formalized or captured within a system of axioms, definitions and theoretical results. We know that much of mathematics is idealized and that, at best, we must, in constructing and using models and applying mathematics, at the very least, often yield to approximations, and yield in the face of harsh realities. Therefore, while the theoretical implications of inconsistency in arithmetic may ultimately be irrelevant, empirical mathematics, the real process of proving theorems or of computation is the actuality of mathematics.

In addition, the acknowledgement of inconsistency as a real possibility resulting from uncertainties is likely to lead to new insights and progress. The notion that inconsistency or ambiguity is necessarily "bad" and incompatible with truth, is just one possibility, and one, indeed, where we might be allowing our fears and anxieties to dominate our thinking rather than opening our minds to new possibilities and new avenues for exploration and progress.

Next, as you might expect, opening the door just a tiny amount with allowing inconsistency propagates like a tidal wave from an earthquake, throughout arithmetic, and has profound (and harsh) theoretical consequences. Since $(-1)^{0.7}$ is absolutely the same number as $(-1)^{70/100}$, we have shown, in accepting an inconsistency if we choose the wrong branch) that there is a number x such that $x = x$ and $x \neq x$. Let us write this latter as $x = x + \alpha$. Then, subtracting $x = x$ we obtain $0 = \alpha$, where α is a non-zero number. But this allows us to divide by α so that we obtain $0 = 1$. Thus, admitting this tiny inconsistency as a result of choosing the wrong branch (which we see becomes highly probable as arithmetic is used in more complicated ways) leads to a veritable storm of inconsistencies throughout mathematics. This devastates our current views of mathematics.

So, just like the ambiguities we have discussed, we must take care of the "paths" of reasoning to avoid such inconsistencies. That is fine with a simple computation or reasoning process such as we have given. But we have demonstrated, unequivocally, that, like ambiguities in taking roots, these alternate paths of reasoning that lead to inconsistency are real possibilities in any mathematical computation or argument. (Taking roots is certainly not the only type of "inverse" problem, in which ambiguity can factor.) It will be profoundly difficult to check that such paths of reasoning are not present in complicated mathematical arguments. This does matter, because it introduces uncertainties about even the most carefully verified proofs of mathematics of any complexity at all.

References

[1] Kleene, S. C. 2009. Introduction to Metamathematics. Ishi Press.